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Applied Mathematics Letters 18 (2005) 1177–1183

Applied Mathematics Letters

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Positive single rupture solutions to a semilinear elliptic equation

Ke Li[a], Hongxia Guo[b], Zongming Guo[c,*]

[a] *School of Mathematics and Statistics, Wuhan University, Wuhan 430072, PR China*
[b] *Department of Mathematics, Zhengzhou University, Zhengzhou 450002, PR China*
[c] *Department of Mathematics, Henan Normal University, Xinxiang 453002, PR China*

Received 26 April 2004; received in revised form 21 December 2004; accepted 1 February 2005

Abstract

The existence of positive single rupture solutions to the semilinear elliptic equation $\Delta u = f(u)$ with $f(0) = \infty$ in a finite ball is obtained via the Pohozaev identity.

MSC: primary 35B50, secondary 35J25

Keywords: Semilinear elliptic equations; Positive single rupture solutions

1. Introduction

We study the existence of positive single rupture solutions to the semilinear elliptic equation

$$\Delta u = f(u) \qquad \text{in } \Omega \setminus \{0\}, \tag{1.1}$$

with

$$\lim_{|x|\to 0} u(x) = 0, \tag{1.2}$$

where $\Omega = B_R = \{x \in \mathbb{R}^n, \ |x| < R\}$ $(n \geq 3)$ is a ball. The nonlinear function $f \in C^1(\mathbb{R}^1 \setminus \{0\})$ satisfies the following condition:

(H) There are $q > 1$, $A > 0$ such that

* Corresponding author.
E-mail addresses: lklikewhj209@tom.com (K. Li), ghx313@163.com (H. Guo), guozm@public.xxptt.ha.cn (Z. Guo).

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doi:10.1016/j.aml.2005.02.006

$(q-1)F(u) \le uf(u)$ for $0 < u \le A$ and $F(A) > 0$, where $F(u) = \int_u^\infty f(s)\,\mathrm{d}s$.

It will be seen later that the condition (H) implies that $u = 0$ is a singular point of f, i.e., $\lim_{u\to 0} f(u) = \infty$.

By a positive single rupture solution U of (1.1) in Ω, we mean that for any compact subset $K \subset \Omega \setminus \{0\}$, $U \in C^2(K)$ satisfies (1.1) in K, $U > 0$ in $\Omega \setminus \{0\}$ and U satisfies (1.2).

The main result of this paper is the following theorem.

Theorem 1.1. *Assume that f satisfies* (H). *Then* (1.1) *has a positive single rupture solution on B_R for some $R \in (0,\infty)$.*

Eq. (1.1) appears in several applications in mechanics and physics, and in particular it can be the equation of equilibrium states in thin films. Equations of the type

$$u_t = -\nabla \cdot (f(u)\nabla\Delta u) - \nabla \cdot (g(u)\nabla u) \tag{1.3}$$

have been used to model the dynamics of thin films of viscose fluids, where $z = u(x,t)$ is the height of the air/liquid interface. The zero set $\Sigma = \{u = 0\}$ is the liquid/solid interface and is sometimes called set of *ruptures*. The coefficient $f(u)$ reflects surface tension effects—a typical choice is $f(u) = u^3$. For background on the equation in (1.3), we refer to [1–7] and the references therein.

In particular, let us assume that $f(u) = u^\ell$, $g(u) = -u^m$, where $\ell, m \in \mathbb{R}$. Then a steady-state equation for (1.3) becomes

$$\Delta u - \frac{u^\kappa}{\kappa} - C = 0, \tag{1.4}$$

where $\kappa = m - \ell + 1$ and C is some constant. (Here we have assumed that $\kappa \ne 0$. If $\kappa = 0$, we have to replace $\frac{u^\kappa}{\kappa}$ by $\log u$.) If we choose $f(u) = u^3$, $g(u) = -u^m$ with $m < 1$, then $\kappa = m - 2 < -1$. Then we get the equation

$$\Delta u = u^{-p}, \qquad p > 1 \tag{1.5}$$

by choosing $C = 0$ and a simple scaling. We easily see that the solutions of (1.5) can explain the behaviours of steady-states for (1.3) in some special cases. The one-dimensional steady-state problem of (1.3) has been studied thoroughly in [5–7] and the references therein. It is seen that the nonlinearity u^{-p} with $p > 1$ satisfies (H) and (1.5) is a special case of (1.1). The existence and properties of solutions of (1.5) in lower dimensions have been studied in [8]. The structure of positive solutions of (1.5) with $-1 < p < 0$ has been studied in [9,10]. Note that, in the latter case, the nonlinearity is non-Lipschitz continuous at $u = 0$. A different type of equation from (1.5)

$$-\Delta u = q(x)u^{-\gamma} \qquad x \in \Omega, \qquad u|_{\partial\Omega} = 0$$

has also been studied in [11] and the references therein.

2. Proof of Theorem 1.1

To prove Theorem 1.1, we consider the initial value problems:

$$\begin{aligned} &u''(r) + \frac{n-1}{r}u'(r) - f(u(r)) = 0, \qquad r > 0,\\ &u(0) = \alpha > 0,\\ &u'(0) = 0. \end{aligned} \tag{2.1}$$

It is known from the theory of ordinary differential equations that, for each $\alpha > 0$, (2.1) has a unique solution $u(r, \alpha)$ in an interval of $[0, \infty)$.

By a simple computation, we have the following Pohozaev identity.

Lemma 2.1. *Let $u(r)$ be a solution of the equation of (2.1) in $(r_1, r_2) \subset (0, \infty)$ and let a be an arbitrary constant. Then for each $r \in (r_1, r_2)$, we have*

$$\frac{\mathrm{d}}{\mathrm{d}r}\left\{r^n\left[\frac{1}{2}u'^2(r) + F(u(r)) + \frac{a}{r}u(r)u'(r)\right]\right\}$$
$$= r^{n-1}\left[nF(u(r)) + af(u(r))u(r) + \left(a + 1 - \frac{n}{2}\right)u'^2(r)\right]. \tag{2.2}$$

Definition 2.2. For each $\alpha \in (0, \infty)$ and $B \geq \alpha$, let $R(\alpha, B)$ be the first $r \in [0, \infty)$ such that $u(r, \alpha) = B$. If there is no such r, we shall adopt the convention that $R(\alpha, B) = \infty$. We also denote that $R_1(\alpha) = R(\alpha, A)$.

Definition 2.3. For q, A, $f(u)$, $F(u)$ given in (H) and $0 < B < A$, letting $\gamma = n-2+2n/(q-1)(>0)$, we define two positive functions $R_*(B)$ and $R^*(B)$ in $(0, A]$ by

$$R_*(B)^2 = 2\gamma\tilde{B}M(\tilde{B})^{-1},$$
$$R^*(B)^2 = 2\left(\frac{n}{q-1}B\right)^2 F(B)^{-1},$$

where $\tilde{B} = \frac{n-2}{2\gamma}B \in (0, B)$, $M(\tilde{B}) = \max_{u\in[\tilde{B},A]} f(u)$.

In the following proposition, we show that for a fixed $0 < B < A$, we can obtain an upper bound and a lower bound for $R(\alpha, B)$. This is crucial in proving Theorem 1.1.

Proposition 2.4. *Assume that f satisfies* (H). *Then for any $0 < B < A$ and $\alpha \in (0, \tilde{B})$, we have:*

$$R_*(B) < R(\alpha, B) < R^*(B), \tag{2.3}$$
$$\frac{q-1}{n}\frac{F(B)}{B}R_*(B) < u'(R(\alpha, B), \alpha) < \frac{2n}{q-1}BR_*(B)^{-1}. \tag{2.4}$$

Proof. In (2.2), letting $u(r) = u(r, \alpha)$, $a = n/(1-q)$ and integrating (2.2) from 0 to r, by (H), we obtain

$$\frac{1}{2}u'^2(r, \alpha) + F(u(r, \alpha)) + \frac{n}{1-q}\frac{u(r, \alpha)u'(r, \alpha)}{r} < 0, \tag{2.5}$$

if $0 < u(r, \alpha) < A$. We claim that (H) implies that $F(u) > 0$ for all $0 < u < A$. Otherwise, there exists $u_1 \in (0, A)$ such that $F(u_1) \leq 0$. Thus,

$$F(u_1) = \int_{u_1}^{\infty} f(s)\,\mathrm{d}s = \int_{u_1}^{A} f(s)\,\mathrm{d}s + \int_{A}^{\infty} f(s)\,\mathrm{d}s = \int_{u_1}^{A} f(s)\,\mathrm{d}s + F(A) \leq 0. \tag{2.6}$$

Since $F(A) > 0$ and $f(A) > 0$ (by (H)), we see that $\int_{u_1}^{A} f(s)\,\mathrm{d}s < 0$ and there exists a maximal point $u_0 \in (u_1, A)$ where f vanishes. It follows from (H) that $F(u_0) \leq 0$. On the other hand, $F(u_0) = F(A) + \int_{u_0}^{A} f(s)\,\mathrm{d}s > 0$. This clearly contradicts $F(u_0) \leq 0$ and our claim holds. This claim and the condition (H) imply that $f(u) > 0$ for $0 < u < A$ and $\lim_{u\to 0} f(u) = \infty$. Therefore, for any $\alpha \in (0, A)$, it follows from (2.5) that $u'(r,\alpha) > 0$ for $r \in (0, R_1(\alpha))$. Furthermore, we have $R_1(\alpha) < \infty$ for all $\alpha \in (0, A)$. Indeed, it is seen from (H) that there is $\tau > 0$ such that

$$f(u) \geq \tau \qquad \text{for } 0 < u \leq A.$$

Thus, we see from (2.1) that

$$r^{n-1}u'(r,\alpha) = \int_0^r s^{n-1} f(u(s,\alpha))\,\mathrm{d}s \geq \frac{\tau}{n} r^n$$

which implies that

$$R_1^2(\alpha) \leq \frac{2n}{\tau}(A - \alpha).$$

On the other hand, we see from (2.5) that for $0 < \alpha < B$,

$$\frac{1}{2}u'^2(R(\alpha, B), \alpha) < \frac{n}{q-1}\frac{B}{R(\alpha, B)}u'(R(\alpha, B), \alpha), \tag{2.7}$$

$$F(B) < \frac{n}{q-1}\frac{B}{R(\alpha, B)}u'(R(\alpha, B), \alpha). \tag{2.8}$$

(2.7) implies that

$$u'(R(\alpha, B), \alpha)R(\alpha, B) < \frac{2n}{q-1}B, \tag{2.9}$$

or

$$u'(R(\alpha, B), \alpha)R(\alpha, B) < (\gamma - n + 2)B. \tag{2.10}$$

From (2.8) and (2.9), we obtain an upper bound for $R(\alpha, B)$:

$$R(\alpha, B)^2 < 2\left(\frac{n}{q-1}B\right)^2 F(B)^{-1} \tag{2.11}$$

for all $\alpha \in (0, B)$. This shows the second inequality of (2.3).

To prove the first inequality of (2.3), there are two cases to consider: (i) $R(\alpha, \tilde{B}) \geq R_*(B)$ and (ii) $R(\alpha, \tilde{B}) < R_*(B)$. For the first case, since $R(\alpha, B) > R(\alpha, \tilde{B})$, we see $R(\alpha, B) > R_*(B)$. For the second case, we need the following comparison argument.

Let $v_\alpha(r) \equiv v(r, \alpha, \tilde{B})$ be the solution of the initial value problem

$$\begin{aligned} &v''(r) + \frac{n-1}{r}v'(r) = \tilde{c}, \qquad \text{for } r > R(\alpha, \tilde{B}) \\ &v(R(\alpha, \tilde{B})) = \tilde{B}, \\ &v'(R(\alpha, \tilde{B})) = u'(R(\alpha, \tilde{B}), \alpha), \end{aligned} \tag{2.12}$$

where $\tilde{c} = M(\tilde{B})$. Then $v_\alpha(r)$ can be written explicitly as

$$\begin{aligned} v_\alpha(r) = \tilde{B} &+ \frac{1}{n-2}\left[\tilde{R}u'(\tilde{R}, \alpha) - \frac{\tilde{c}}{n}\tilde{R}^2\right] \\ &- \frac{1}{n-2}\left[\tilde{R}^{n-1}u'(\tilde{R}, \alpha) - \frac{\tilde{c}}{n}\tilde{R}^n\right]r^{2-n} + \frac{\tilde{c}}{2n}r^2 - \frac{\tilde{c}}{2n}\tilde{R}^2, \end{aligned} \tag{2.13}$$

where $\tilde{R} = R(\alpha, \tilde{B})$. Thus,

$$\begin{aligned} v_\alpha(r) &\le \tilde{B} + \frac{1}{n-2}\tilde{R}u'(\tilde{R}, \alpha) + \frac{\tilde{c}}{n(n-2)}\tilde{R}^n r^{2-n} + \frac{\tilde{c}}{2n}r^2 \\ &< \frac{1}{n-2}\left[\gamma \tilde{B} + \frac{\tilde{c}}{2}r^2\right] \\ &= B - \frac{1}{n-2}\left[\gamma \tilde{B} - \frac{\tilde{c}}{2}r^2\right] \\ &\le B \end{aligned}$$

for all $r \in [R(\alpha, \tilde{B}), R_*(B)]$. Therefore, (2.3) follows if we can show that $u(r, \alpha) \le v_\alpha(r)$ on $[R(\alpha, \tilde{B}), R_*(B)]$. Let $w(r) = v_\alpha(r) - u(r, \alpha)$. Then

$$(r^{n-1}w'(r))' = r^{n-1}(\tilde{c} - f(u(r))) \ge 0 \tag{2.14}$$

as long as $\tilde{B} \le u \le A$. Moreover,

$$w(R(\alpha, \tilde{B})) = 0 = w'(R(\alpha, \tilde{B})). \tag{2.15}$$

Integrating (2.14) from $\tilde{R}$ to r twice and using (2.15), we obtain

$$u(r, \alpha) \le v_\alpha(r) \le B \qquad \text{for } r \in [R(\alpha, \tilde{B}), R_*(B)].$$

This implies that the first inequality of (2.3) holds. (2.4) follows from (2.3), (2.8) and (2.9). This completes the proof of Proposition 2.4. □

An immediate consequence of the estimates in (2.3) and (2.4) is the following existence result for (1.1) and (1.2).

Lemma 2.5. *Assume that f satisfies* (H). *If $\{\alpha_k\}$ is a sequence with $\lim_{k\to\infty} \alpha_k = 0$ and $\lim_{k\to\infty} R(\alpha_k, A) = R \le \infty$, then there is a subsequence $\{\alpha'_k\}$ of $\{\alpha_k\}$ and a nonnegative function U such that $u(\cdot, \alpha'_k)$ converges to U pointwise in $(0, R)$ and also in $C^2[a, b]$ for $[a, b] \subset (0, R)$. Moreover, U is a positive single rupture solution of the equation of* (2.1) *in B_R.*

Proof. We first claim that for any compact subinterval $[a, b] \subset (0, R)$, $u^{(j)}(\cdot, \alpha_k)$ are uniformly bounded on $[a, b]$, for $j = 0, 1, 2, 3$. We easily know that

$$\lim_{B\to 0} (R^*(B))^2 = \lim_{B\to 0} 2(nB/(q-1))^2 F(B)^{-1} = 0. \tag{2.16}$$

Therefore, there is a $0 < B < A$ such that $R^*(B) \le a$. Proposition 2.4 and the monotonicity of $u(\cdot, \alpha_k)$ imply that $R(\alpha_k, B) \le a$ and $u(a, \alpha_k) \ge B$ for large k. On the other hand, using our assumption $R(\alpha_k, A) \to R$ as $k \to \infty$, we have $b < R(\alpha_k, A)$ for large k, i.e., $u(b, \alpha_k) \le u(R(\alpha_k, A), \alpha_k) = A$ for large k. Thus, we obtain that

$$B \le u(r, \alpha_k) \le A \qquad \text{in } [a, b] \text{ for large } k. \tag{2.17}$$

Now for any solution $u(r)$ of (2.1) and $0 < r_1 < r_2 < R$, $u(r)$ satisfies:

$$u'(r_2)r_2^{n-1} = u'(r_1)r_1^{n-1} + \int_{r_1}^{r_2} t^{n-1} f(u(t))\,\mathrm{d}t. \tag{2.18}$$

Choosing $r_1 = R(\alpha_k, B)$, $r_2 = r \in (R(\alpha_k, B), b]$ and $u(r) = u(r, \alpha_k)$ in (2.18) and using (2.17), we obtain

$$|u'(r, \alpha_k)| \le \frac{1}{r} u'(R(\alpha_k, B), \alpha_k) R(\alpha_k, B) \left\{ \frac{R(\alpha_k, B)}{r} \right\}^{n-2} + \frac{M(B)}{n} r. \tag{2.19}$$

By (2.3) and (2.9), there is a positive constant $c = c(a, b, B) > 0$ such that:

$$|u'(r, \alpha_k)| \le c \qquad \text{in } [a, b] \text{ for large } k. \tag{2.20}$$

Hence, by the equation in (2.1) and (2.20), we can easily obtain that $u^{(j)}(\cdot, \alpha_k)$ are uniformly bounded on $[a, b]$, for $j = 0, 1, 2, 3$.

Now, the Ascoli–Arzela theorem and the diagonal process imply that there are a subsequence $\{\alpha_k'\}$ of $\{\alpha_k\}$ and a C^2 nonnegative function $U(r)$ such that $u(\cdot, \alpha_k')$ converges to $U(r)$ pointwise in $(0, R)$ and also in $C^2[a, b]$ for any compact subinterval $[a, b]$ of $(0, R)$. Since $f(0) = \infty$, we see that $U \not\equiv 0$ in $(0, R)$. Moreover, the strong maximum principle implies that $U > 0$ in $(0, R)$. It is clear that $U(r)$ satisfies (2.1) in $(0, R)$. Now we claim that (2.3) implies that $U(r)$ tends to 0 as $r \to 0^+$. For any $B < A$, if $r \le R_*(B)$, then $u(r, \alpha_k') \le B$, which implies that $U(r) \le B$ in $[0, R_*(B)]$. Since $R_*(B) \to 0$ as $B \to 0$, we have $U(r) \to 0$ as $r \to 0^+$. Therefore, $U(r)$ is a positive single rupture solution of (2.1) in B_R. This completes the proof of Lemma 2.5. $\square$

We are now ready to complete the proof of Theorem 1.1.

Proof of Theorem 1.1. Let $\{\alpha_k\}$ be a sequence with $\lim_{k\to\infty} \alpha_k = 0$. Since $R_1^2(\alpha_k) \le \frac{2n}{\tau}(A - \alpha_k)$, we can assume that $\lim_{k\to\infty} R(\alpha_k, A) = R < \infty$. By arguments similar to those in the proof of Lemma 2.5, we see that there is a positive single rupture solution U of (1.1) in B_R. $\square$

Acknowledgement

The authors would like to thank the referee for his/her valuable suggestions.

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